Linear Algebra (Spring 2018)
Topic: Theorem of simultaneous diagonalizability [1]
Amin – 20 May 2018

Theorem: Let $A, B \in M_n(\mathbb{C})$ be *diagonalizable*. Then $AB = BA$ if and only if there is an invertible $S \in M_n(\mathbb{C})$ such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal (For the case of $A, B \in M_n(\mathbb{R})$, it can be easily proved just as the same approach for complex field).

Proof:

Let $A, B \in M_n(\mathbb{C})$ be diagonalizable.
($\Leftarrow$) Suppose there is an invertible $S \in M_n(\mathbb{C})$ such that A, B are simultaneously diagonalizable in which $S^{-1}AS$ and $S^{-1}BS$ are both diagonal. Since diagonal matrices commute, then we have

$$S^{-1}ASS^{-1}BS = S^{-1}BSS^{-1}AS,$$

which implies $AB = BA$.
($\Rightarrow$) Suppose $AB = BA$. Since A is diagonalizable, there exists a nonsingular matrix Q such that

$$D = Q^{-1}AQ = \begin{bmatrix} \lambda_1 I_{n_1} & & & 0 \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ 0 & & & \lambda_p I_{n_p} \end{bmatrix}$$

where $\lambda_1, \dots \lambda_p$ are the distinct eigenvalues of A and $n_1, \dots, n_p$ are their multiplicities, respectively (here, the repeated eigenvalues are grouped together). Assume this similarity transformation does not diagonalize B, where $B^{'} := Q^{-1}BQ$ is not a diagonal matrix. Since A, B commutes we have

$$DB^{'} = Q^{-1}AQQ^{-1}BQ = Q^{-1}BQQ^{-1}AQ = B^{'}D.$$

Since $B^{'}$ commutes with D, then $B^{'}$ must be a block diagonal matrix conformal to D and hence

$$B^{'} = \begin{bmatrix} B^{'}_{n_1} & & & 0 \\ & B^{'}_{n_2} & & \\ & & \ddots & \\ 0 & & & B^{'}_{n_p} \end{bmatrix},$$

where $B^{'}_{n_i} \in M_{n_i}$. Since B is diagonalizable, then there exists a nonsingular matrix P such that $E := P^{-1}BP$ which is a diagonal matrix. Then, $B^{'} = Q^{-1}BQ = Q^{-1}PEP^{-1}Q = Y^{-1}EY$ where $Y := P^{-1}Q$ is an invertible matrix. Therefore, $B^{'}$ is diagonalizable and this further implies that $B^{'}_{n_i}, i = 1, \dots, n_p$ is diagonalizable (by Claim 1). Then, there exists nonsingular $X_i \in M_{n_i}$ such that $X^{-1}_{n_i} B^{'}_{n_i} X_{n_i}$ for $i = 1, \dots, n_p$ is a diagonal matrix. Let $X := diag\{X_{n_1}, X_{n_2}, \dots, X_{n_p}\}$ (a block diagonal matrix). This implies $\Lambda = X^{-1}B^{'}X$ and $\Gamma = X^{-1}DX$ are diagonal matrices. Therefore, there exists invertible $S := QX$ such that $\Lambda = S^{-1}BS$ and $\Gamma = S^{-1}AS$ are diagonal matrices and the proof is completed. $\square$

▲ *Claim 5: Define $A \in M_n(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ such that*

$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix},$$

where $A_k \in M_{n_i}$ are block matrices. Then, A is diagonalizable if and only if each of A_k is diagonalizable.
Proof:
($\Leftarrow$) Let $A_1, \ldots, A_k$ be diagonalizable. Then, there exist invertible Q_i such that $Q_i^{-1} A_i Q_i$ is diagonal matrix. Let $Q \coloneqq Q_1 \oplus \ldots \oplus Q_k$ and then clearly we have $D = Q^{-1}AQ$ is a diagonal matrix. Hence, A is diagonalizable.
($\Rightarrow$) Let A is diagonalizable, then there exist invertible Q such that $D = Q^{-1}AQ$ where $D = diag\{\lambda_1, \ldots, \lambda_k\}$ is a diagonal matrix. Then, $AQ = QD$ and we have

$$\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix} \begin{bmatrix} x_1^j \\ \vdots \\ x_k^j \end{bmatrix} = \lambda_j \begin{bmatrix} x_1^j \\ \vdots \\ x_k^j \end{bmatrix},$$

where $v_j = \left[x_1^j, \ldots, x_k^j\right]^t$ is the j-th column of Q and x_ℓ^j is a vector as well such that $A_\ell x_\ell^j = \lambda_j x_\ell^j$ where $\ell = 1, \ldots, k$ and A_ℓ is $n_\ell \times n_\ell$ matrix. Knowing that Q is invertible, we can define the invertible submatrix (and full rank) as $Y \coloneqq \left[x_\ell^{m_1}, \ldots, x_\ell^{m_{n_\ell}}\right]$ which is an $n_\ell \times n_\ell$ matrix where $m_1, \ldots, m_{n_\ell} \in \{1, 2, \ldots, k\}$. Then, we can write $A_\ell Y = Y\Lambda$ where $\Lambda = diag\{\lambda_{m_1}, \ldots, \lambda_{m_\ell}\}$ which is a diagonal matrix. Hence, $A_\ell = Y\Lambda Y^{-1}$ and the proof is completed. □

References

[1] R. A. Horn and C. R. Johnson, *Matrix Analysis*. New York, NY, USA: Cambridge University Press, 2nd ed., 2012.